



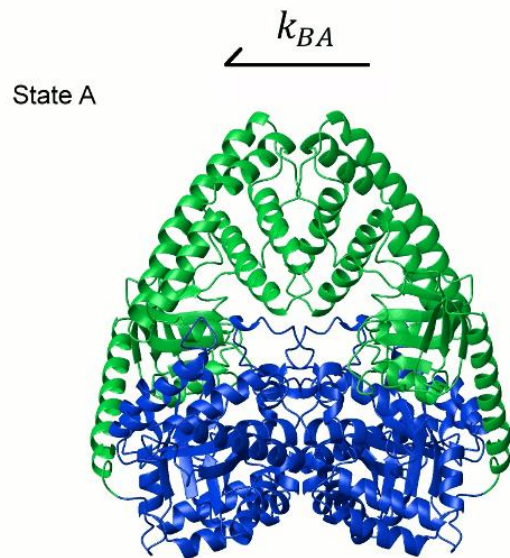
# Model of Large Scale Conformational Mobility in Proteins

Yaroslav Ryabov

# Outline

- Conformational transitions on different time scales
- Equation of rotational diffusion
- Eigenfunctions of free diffusion Liouville operator
- Conformational transitions and associated system of linear differential equations
- Eigen and non-Eigen decompositions
- Abel's impossibility and the way we trick it:  
Time domain vs. Frequency domain.
- What to expect: Illustrative calculations

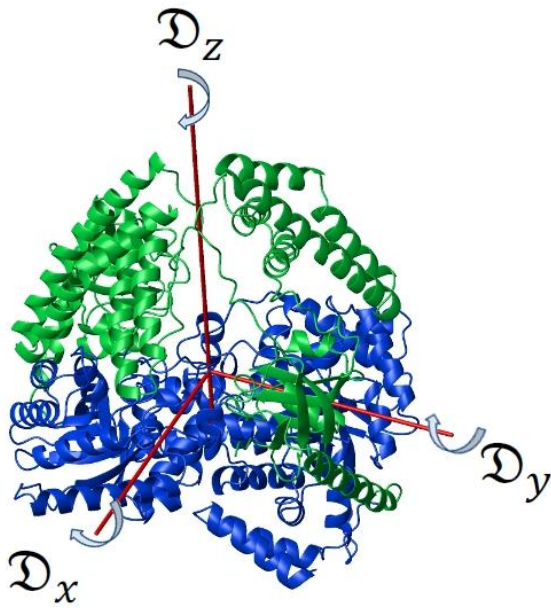
## Time scale of Conformational transitions



$$\tau_c = \frac{1}{k_{AB} + k_{BA}}$$

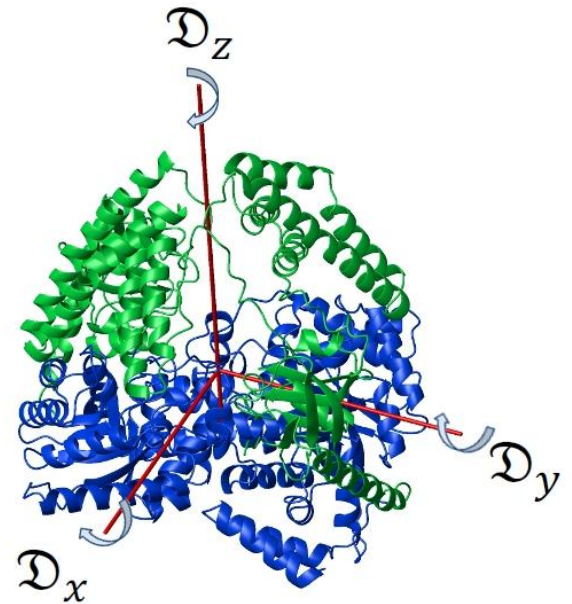
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## Time scale of Rotational diffusion



$$\tau_{\mathcal{D}} = \frac{1}{2(\mathcal{D}_x + \mathcal{D}_y + \mathcal{D}_z)}$$

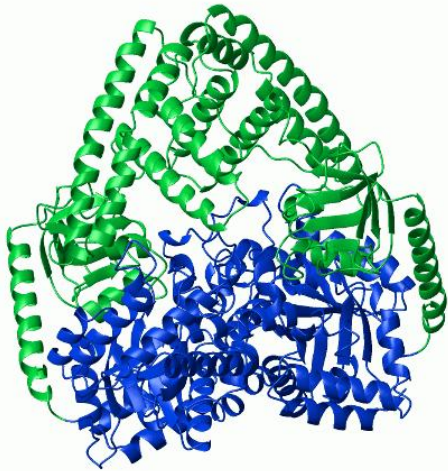
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# Conformation transitions on different time scales

## Slow Exchange

State A



$$\tau_c \gg \tau_D$$

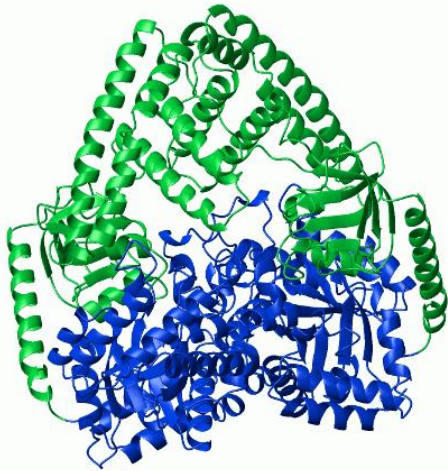
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Approximation of **two species**  
with **two different structures**  
and **two diffusion tensors**

$$\mathcal{D}^A \text{ and } \mathcal{D}^B$$

# Conformation transitions on different time scales

State A



## Fast Exchange

$$\tau_c \ll \tau_D$$

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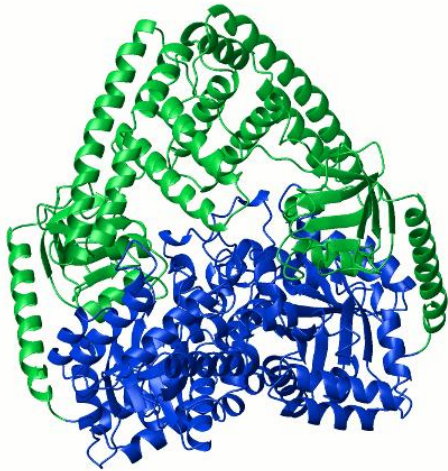
Approximation of **single conformer**  
with **one averaged structure**  
and **one averaged diffusion tensor**

$\bar{D}$

# Conformation transitions on different time scales

## Intermediate Exchange

State A



$$\tau_c \sim \tau_D$$

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# Rotational diffusion of Rigid molecule

## Orientation correlation function

$$C_l(t) = \langle P_l[\mathbf{n}(t) \cdot \mathbf{n}(0)] \rangle$$

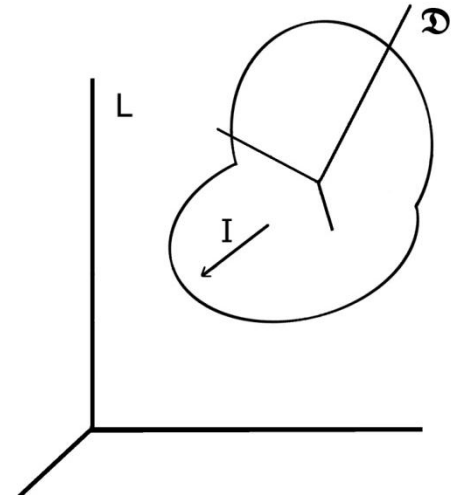
Fourier transform

Spectral density

$$J_l(\omega)$$

Linear Combination

Experimental observables: R1, R2 etc.





# Rotational diffusion of Rigid molecule

## Orientation correlation function

$$C_l(t) = \langle P_l[\mathbf{n}(t) \cdot \mathbf{n}(0)] \rangle$$

Spectral density

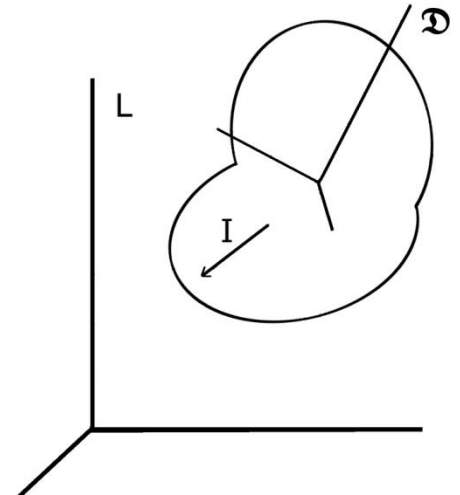
$$J_l(\omega)$$

## Statistic averaging

$$\langle \dots \rangle = \iint \dots \underline{p(\Omega, t | \Omega^0)} p_{eq}(\Omega^0) d\Omega d\Omega^0$$

$$p_{eq}(\Omega^0) = \frac{1}{8\pi^2}$$

Isotropic environment



## Rotational diffusion of Rigid molecule

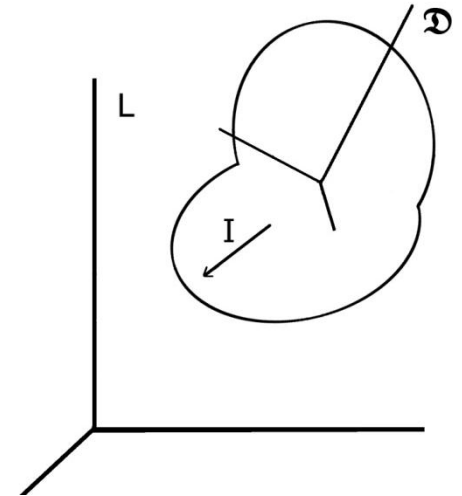
### Equation for Green's function

$$\frac{\partial p(\Omega, t | \Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D} \hat{L} p(\Omega, t | \Omega^0)$$

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### Initial conditions

$$p(\Omega, t | \Omega^0) |_{t=+0} = \delta(\Omega - \Omega^0)$$



### Operator of spatial rotations

$$\hat{L}^T = \{L_x, L_y, L_z\}$$

$$\hat{L} = i[\vec{r} \times \vec{\nabla}] \quad \vec{\nabla} = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

## Rotational diffusion of Rigid molecule

### Equation for Green's function

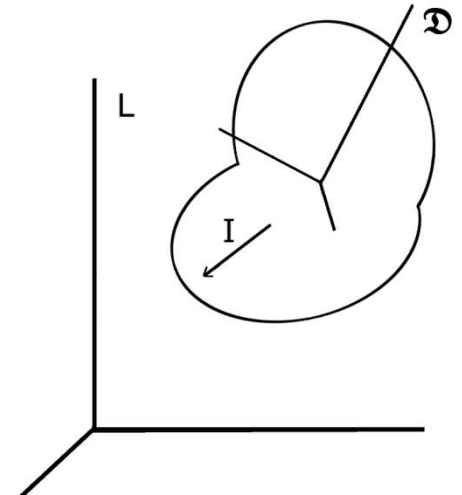
$$\frac{\partial p(\Omega, t | \Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D} \hat{L} p(\Omega, t | \Omega^0)$$

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Eigen functions  $\Psi_{mn}^l(\Omega)$  of  $\hat{L}^T \mathfrak{D} \hat{L}$  operator

$$\hat{L}^T \mathfrak{D} \hat{L} \Psi_{mn}^l(\Omega) = E_n^l \Psi_{mn}^l(\Omega)$$

$$p(\Omega, t | \Omega^0) = \sum_{l=0}^{\infty} \sum_{m,n=-l}^l c_{mn}^l(t) \Psi_{mn}^l(\Omega)$$



Ansatz Solution

# Rotational diffusion of Rigid molecule

## Equation for Green's function

$$\frac{\partial p(\Omega, t | \Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D} \hat{L} p(\Omega, t | \Omega^0)$$

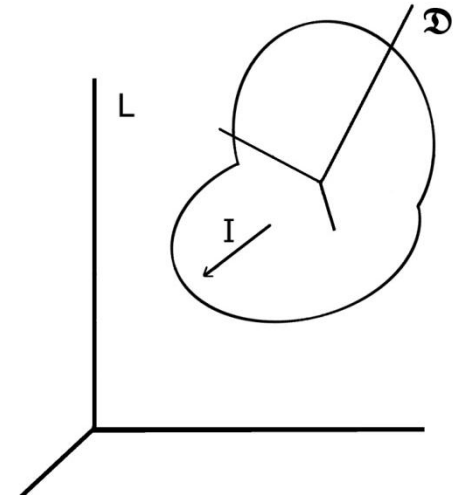

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Eigen functions  $\Psi_{mn}^l(\Omega)$  of  $\hat{L}^T \mathfrak{D} \hat{L}$  operator

$$\hat{L}^T \mathfrak{D} \hat{L} \Psi_{mn}^l(\Omega) = E_n^l \Psi_{mn}^l(\Omega)$$

$$p(\Omega, t | \Omega^0) = \sum_{l=0}^{\infty} \sum_{m,n=-l}^l \Psi_{mn}^{l*}(\Omega^0) \Psi_{mn}^l(\Omega) e^{-E_n^l t}$$

$$\Psi_{mn}^l(\Omega) = \sum_{k=-l}^l A_{mk}^l(\mathfrak{D}_x, \mathfrak{D}_y, \mathfrak{D}_z) \underline{D_{kn}^{(l)}(\Omega)}$$



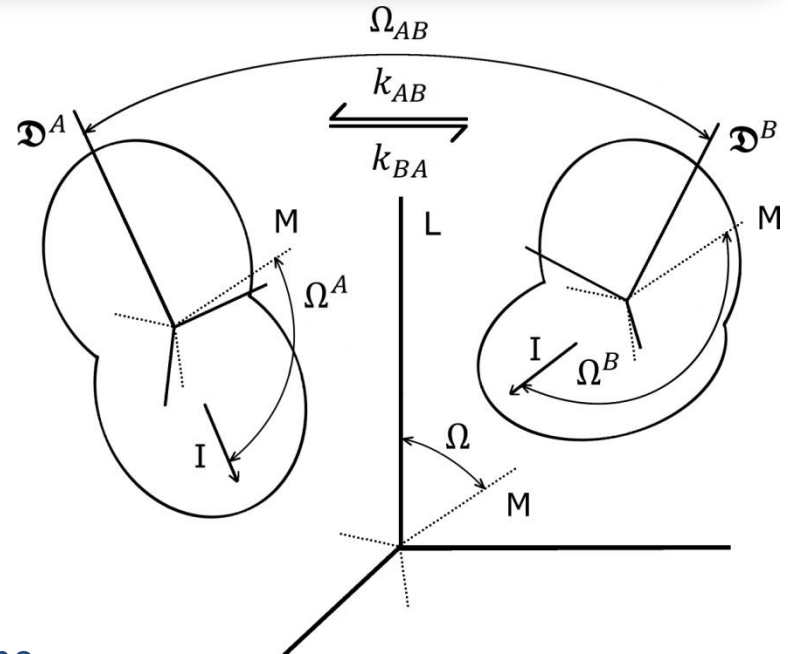
Solution

Wigner rotation matrix

## Rotational diffusion of semi-Rigid molecule

### Conformation transition between discrete set of states

- Molecule tumbles in isotropic solvent
- Molecule exchanges between discrete conformations  $\varepsilon = A, B, \dots$
- In each conformation state molecule is rigid and have diffusion tensor  $\mathfrak{D}^\varepsilon$
- The transition time is much shorter than the time which molecule spends in any conformation



# Rotational diffusion of semi-Rigid molecule

## Equations for Green's functions

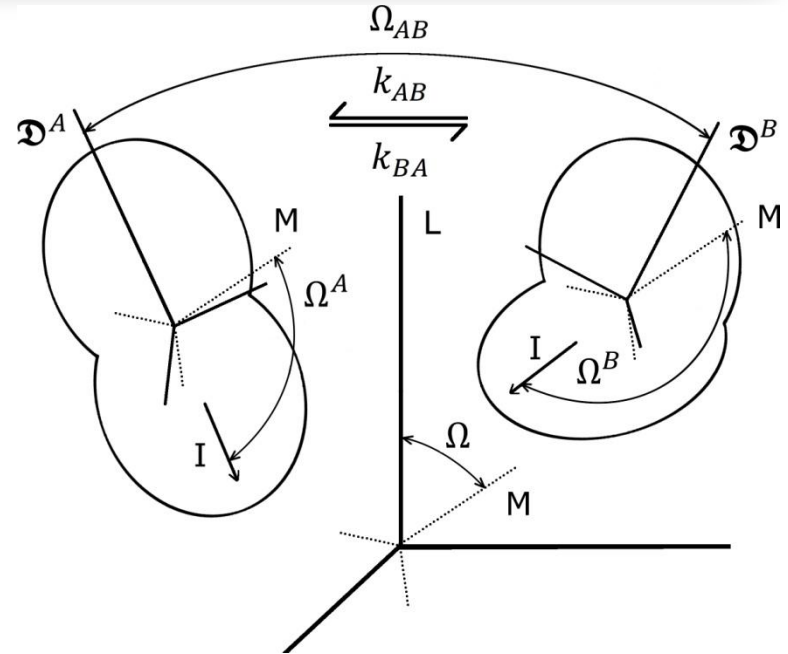
$$\frac{\partial p^{\varepsilon\eta}(\Omega, t|\Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D}^\varepsilon \hat{L} p^{\varepsilon\eta}(\Omega, t|\Omega^0)$$

$$+ \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} p^{\mu\eta}(\Omega, t|\Omega^0)$$

Conformation exchange

## Initial conditions

$$p^{\varepsilon\eta}(\Omega, t|\Omega^0)|_{t=+0} = \delta_{\varepsilon\eta} \delta(\Omega - \Omega^0)$$

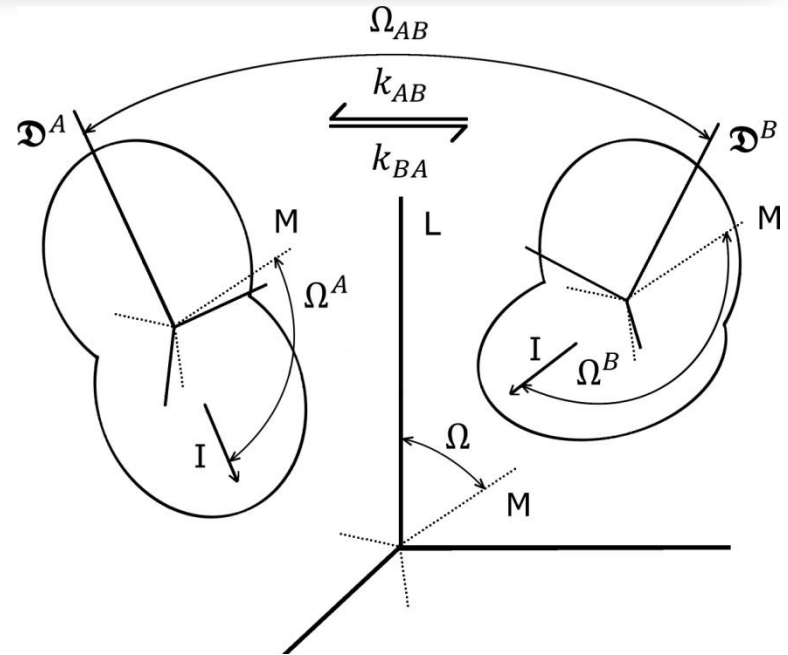


$$\mathbf{K} = \begin{pmatrix} -\sum_{\varepsilon \neq A} k_{\varepsilon A} & k_{AB} & \vdots \\ k_{BA} & -\sum_{\varepsilon \neq B} k_{\varepsilon B} & \vdots \\ \dots & \dots & \dots \end{pmatrix}$$

## Rotational diffusion of semi-Rigid molecule

### Equations for Green's functions

$$\frac{\partial p^{\varepsilon\eta}(\Omega, t|\Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D}^\varepsilon \hat{L} p^{\varepsilon\eta}(\Omega, t|\Omega^0) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} p^{\mu\eta}(\Omega, t|\Omega^0)$$



Eigenfunctions are not available  
in closed forms

## Rotational diffusion of semi-Rigid molecule

### Equations for Green's functions

$$\frac{\partial p^{\varepsilon\eta}(\Omega, t|\Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D}^\varepsilon \hat{L} p^{\varepsilon\eta}(\Omega, t|\Omega^0) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} p^{\mu\eta}(\Omega, t|\Omega^0)$$

### Ansatz of non-eigen decomposition

$$p^{\varepsilon\eta}(\Omega, t|\Omega^0, 0) = \sum_{l=0}^{\infty} \sum_{g,q=-l}^l c_{l,gq}^{\varepsilon\eta}(t) \underline{D_{gq}^{(l)}(\Omega)}$$

$$\Psi_{mn}^{\varepsilon,l}(\Omega) = \sum_{k,p=-l}^l A_{mp}^{\varepsilon,l} D_{pk}^{(l)}(\Omega^\varepsilon) \underline{D_{kn}^{(l)}(\Omega)}$$

Wigner rotation matrix



## Rotational diffusion of semi-Rigid molecule

### Equations for Green's functions


$$\frac{\partial p^{\varepsilon\eta}(\Omega, t|\Omega^0)}{\partial t} = -\hat{L}^T \mathfrak{D}^\varepsilon \hat{L} p^{\varepsilon\eta}(\Omega, t|\Omega^0) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} p^{\mu\eta}(\Omega, t|\Omega^0)$$

### Ansatz of non-eigen decomposition

$$p^{\varepsilon\eta}(\Omega, t|\Omega^0, 0) = \sum_{l=0}^{\infty} \sum_{g,q=-l}^l c_{l,gq}^{\varepsilon\eta}(t) D_{gq}^{(l)}(\Omega)$$

$$U_{pk}^{\varepsilon,l} = \sum_{n=-l}^l D_{np}^{(l)*}(\Omega) \Psi_{nk}^{\varepsilon,l}(\Omega)$$

Unitary transformation between  
sets of  $\Psi_{nk}^{\varepsilon,l}(\Omega)$   
and  $D_{kn}^{(l)}(\Omega)$



## Rotational diffusion of semi-Rigid molecule

### Equations for decomposition coefficients

$$\frac{\partial c_{l,mn}^{\varepsilon\eta}(t)}{\partial t} = - \sum_{s,q=-l}^l U_{ns}^{\varepsilon,l} E_s^{\varepsilon,l} U_{sq}^{\varepsilon,l\dagger} c_{l,mq}^{\varepsilon\eta}(t) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} c_{l,mn}^{\mu\eta}(t)$$

### Initial conditions

$$c_{l,mn}^{\varepsilon\eta}(t) \Big|_{t=+0} = \frac{2l+1}{8\pi^2} D_{mn}^{(l)*}(\Omega^0) \delta_{\varepsilon\eta}$$

Formal Solution

$$p^{\varepsilon\eta}(\Omega, t | \Omega^0, 0) = \sum_{l=0}^{\infty} \sum_{g,q=-l}^l c_{l,gq}^{\varepsilon\eta}(t) D_{gq}^{(l)}(\Omega)$$

## Rotational diffusion of semi-Rigid molecule

### Equations for decomposition coefficients

$$\frac{\partial c_{l,mn}^{\varepsilon\eta}(t)}{\partial t} = - \sum_{s,q=-l}^l U_{ns}^{\varepsilon,l} E_s^{\varepsilon,l} U_{sq}^{\varepsilon,l\dagger} c_{l,mq}^{\varepsilon\eta}(t) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} c_{l,mn}^{\mu\eta}(t)$$

Ryabov, Clore, Schwieters (2012)

## Rotational diffusion of semi-Rigid molecule

### Equations for decomposition coefficients (no restrictions)

$$\frac{\partial c_{l,mn}^{\varepsilon\eta}(t)}{\partial t} = - \sum_{s,q=-l}^l U_{ns}^{\varepsilon,l} E_s^{\varepsilon,l} U_{sq}^{\varepsilon,l\dagger} c_{l,mq}^{\varepsilon\eta}(t) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} c_{l,mn}^{\mu\eta}(t)$$

Ryabov, Clore, Schwieters (2012)

### Equations for decomposition coefficients (axial symmetry and co-linearity of the axes of symmetry )

$$\frac{\partial f_{l,m}^{\varepsilon\eta}(t)}{\partial t} = -[l(l+1)\mathcal{D}_{\perp}^{\varepsilon} + m^2(\mathcal{D}_{\parallel}^{\varepsilon} - \mathcal{D}_{\perp}^{\varepsilon})]f_{l,m}^{\varepsilon\eta}(t) + \sum_{\mu \neq \varepsilon} K_{\varepsilon\mu} f_{l,m}^{\mu\eta}(t)$$

Berne, Pecora (1968); Wong, Case and Szabo (2009)

# Rotational diffusion of semi-Rigid molecule

## Equations for decomposition coefficients

$$\frac{\partial \mathbf{c}_{l,m}^\eta(t)}{\partial t} = \mathbf{M}_l \mathbf{c}_{l,m}^\eta(t)$$

Matrix Form

$$k^\varepsilon = \sum_{\varepsilon \neq \eta} k_{\varepsilon\eta}$$

$$\mathbf{M}_l = \begin{pmatrix} \mathbf{U}^{A,l} \mathbf{E}^{A,l} \mathbf{U}^{A,l\dagger} - k^A \mathbf{I}_{2l+1} & k_{AB} \mathbf{I}_{2l+1} & \vdots \\ k_{BA} \mathbf{I}_{2l+1} & \mathbf{U}^{B,l} \mathbf{E}^{B,l} \mathbf{U}^{B,l\dagger} - k^B \mathbf{I}_{2l+1} & \vdots \\ \dots & \dots & \dots \end{pmatrix}$$

$$[\mathbf{c}_{l,m}^\eta(t)]_\varepsilon = \begin{pmatrix} c_{l,m}^{\varepsilon\eta}(t) \\ \vdots \\ c_{l,m-l}^{\varepsilon\eta}(t) \end{pmatrix} \quad \mathbf{E}^{\varepsilon,l} = \begin{pmatrix} E_l^{\varepsilon,l} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & E_{-l}^{\varepsilon,l} \end{pmatrix}$$

# Rotational diffusion of semi-Rigid molecule

## Equations for decomposition coefficients

$$\frac{\partial \mathbf{c}_{l,m}^\eta(t)}{\partial t} = \mathbf{M}_l \mathbf{c}_{l,m}^\eta(t)$$

Block dimension  $(2l + 1)$

$$\mathbf{M}_l = \begin{pmatrix} \mathbf{U}^{A,l} \mathbf{E}^{A,l} \mathbf{U}^{A,l\dagger} - k^A \mathbf{I}_{2l+1} & k_{AB} \mathbf{I}_{2l+1} & \vdots \\ k_{BA} \mathbf{I}_{2l+1} & \mathbf{U}^{B,l} \mathbf{E}^{B,l} \mathbf{U}^{B,l\dagger} - k^B \mathbf{I}_{2l+1} & \vdots \\ \dots & \dots & \dots \end{pmatrix}$$

$N_{states}$

## Rotational diffusion of semi-Rigid molecule

### Equations for decomposition coefficients

$$\frac{\partial \mathbf{c}_{l,m}^{\eta}(t)}{\partial t} = \mathbf{M}_l \mathbf{c}_{l,m}^{\eta}(t)$$

---

$\mathbf{M}_l$  dimension  $N_{states}(2l + 1)$

$l = 1$  Dielectric spectroscopy

$l = 2$  NMR, light scattering

## Rotational diffusion of semi-Rigid molecule

### Equations for decomposition coefficients

$$\frac{\partial \mathbf{c}_{l,m}^{\eta}(t)}{\partial t} = \mathbf{M}_l \mathbf{c}_{l,m}^{\eta}(t)$$

$\mathbf{M}_l$  dimension  $N_{states}(2l + 1)$

No solution in closed forms for  $\mathbf{c}_{l,m}^{\eta}(t)$

Even for  $l = 1$  and  $N_{states} = 2$

due to **Abel impossibility theorem (1824)**



## Rotational diffusion of semi-Rigid molecule

Equations for decomposition coefficients  
in frequency domain

$$\sigma \tilde{\mathbf{c}}_{l,m}^{\eta}(\sigma) - \mathbf{c}_{l,mn}^{\eta}(t)|_{t=+0} = \mathbf{M}_l \tilde{\mathbf{c}}_{l,m}^{\eta}(\sigma)$$

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$$\sigma = \gamma + i\omega$$

Complex frequency

$$\tilde{\mathbf{c}}_{l,m}^{\eta}(\sigma) := \mathcal{L}[\mathbf{c}_{l,m}^{\eta}(t)]$$

Laplace transform

$$\left[ \mathbf{c}_{l,mn}^{\eta}(t) \Big|_{t=+0} \right]_{\varepsilon} = \delta_{\varepsilon\eta} \frac{2l+1}{8\pi^2} \begin{pmatrix} D_{m,l}^{(l)*}(\Omega^0) \\ \vdots \\ D_{m,-l}^{(l)*}(\Omega^0) \end{pmatrix}$$

Initial conditions

## Rotational diffusion of semi-Rigid molecule

Equations for decomposition coefficients  
in frequency domain

$$\tilde{\mathbf{c}}_{l,m}^{\eta}(\sigma) = (\sigma \mathbf{I}_{N_{states}(2l+1)} - \mathbf{M}_l)^{-1} \mathbf{c}_{l,mn}^{\eta}(t) \Big|_{t=+0}$$

$$\left( \begin{array}{cc|c} \mathbf{U}^{A,l} \mathbf{Q}^{A,l} \mathbf{U}^{A,l\dagger} & k_{AB} \mathbf{I}_{2l+1} & \vdots \\ k_{BA} \mathbf{I}_{2l+1} & \mathbf{U}^{B,l} \mathbf{Q}^{B,l} \mathbf{U}^{B,l\dagger} & \vdots \\ \dots & \dots & \dots \end{array} \right)^{-1}$$

Matrix inversion does **NOT**  
depend on solving Eigen problem

**No restrictions from Abel theorem**

$$\mathbf{Q}^{\varepsilon,l} = \mathbf{E}^{\varepsilon,l} + (k^{\varepsilon} + \sigma) \mathbf{I}_{2l+1}$$

## Rotational diffusion of semi-Rigid molecule

### Green's functions in frequency domain

$$\tilde{p}^{\varepsilon\eta}(\Omega, \sigma | \Omega^0, 0) = \sum_{l=0}^{\infty} \sum_{g,q=-l}^l \tilde{c}_{l,gq}^{\varepsilon\eta}(\sigma) D_{gq}^{(l)}(\Omega)$$

### Green's function in time domain

$$p^{\varepsilon\eta}(\Omega, t | \Omega^0, 0) = \sum_{l=0}^{\infty} \sum_{g,q=-l}^l c_{l,gq}^{\varepsilon\eta}(t) D_{gq}^{(l)}(\Omega)$$

## Rotational diffusion of semi-Rigid molecule

### Correlation function in frequency domain

$$\tilde{C}(\sigma) = \iint \underbrace{\tilde{p}(\Omega_{\text{LM}}, \sigma | \Omega_{\text{LM}}^0, 0)}_{\text{red underline}} p_{\text{eq}}(\Omega_{\text{LM}}^0) \times \\ \times \sum_{m, k, k' = -l}^l D_{mk}^{l*}(\Omega_{\text{LM}}) D_{mk'}^l(\Omega_{\text{LM}}^0) D_{k0}^{l*}(\Omega_{\text{MI}}) D_{k'0}^l(\Omega_{\text{MI}}) d\Omega_{\text{LM}} d\Omega_{\text{LM}}^0$$

### Correlation function in time domain

$$C(t) = \iint \underbrace{p(\Omega_{\text{LM}}, t | \Omega_{\text{LM}}^0, 0)}_{\text{red underline}} p_{\text{eq}}(\Omega_{\text{LM}}^0) \times \\ \times \sum_{m, k, k' = -l}^l D_{mk}^{l*}(\Omega_{\text{LM}}) D_{mk'}^l(\Omega_{\text{LM}}^0) D_{k0}^{l*}(\Omega_{\text{MI}}) D_{k'0}^l(\Omega_{\text{MI}}) d\Omega_{\text{LM}} d\Omega_{\text{LM}}^0$$

## Rotational diffusion of semi-Rigid molecule

Correlation function in frequency domain (no restrictions)

Ryabov, Clore, Schwieters (2012)

$$\tilde{C}_l(\sigma) = \frac{4\pi}{2l+1} \sum_{\varepsilon, \eta} \mathbf{Y}_l^T(\Omega_{\varepsilon I}) \mathbf{A}^{\varepsilon, l \dagger} \mathbf{R}^{\varepsilon \eta}(\sigma) \mathbf{A}^{\eta, l} \mathbf{Y}_l^*(\Omega_{\eta I}) P_{eq}^{\eta}$$

Correlation function in time domain (axial symmetry and co-linearity of the axes of symmetry )

$$C_l(t) = \frac{4\pi}{2l+1} \sum_{m=-l}^l \sum_{\varepsilon, \eta} f_{l,m}^{\varepsilon \eta}(t) Y_{l,m}(\Omega^{\varepsilon}) Y_{l,m}^*(\Omega^{\eta}) P_{eq}^{\eta}$$

Berne, Pecora (1968); Wong, Case and Szabo (2009)

## Rotational diffusion of semi-Rigid molecule

Correlation function in frequency domain (no restrictions)

$$\tilde{C}_l(\sigma) = \frac{4\pi}{2l+1} \sum_{\varepsilon, \eta} \mathbf{Y}_l^T(\Omega_{\varepsilon I}) \mathbf{A}^{\varepsilon, l \dagger} \mathbf{R}^{\varepsilon \eta}(\sigma) \mathbf{A}^{\eta, l} \mathbf{Y}_l^*(\Omega_{\eta I}) P_{eq}^{\eta}$$

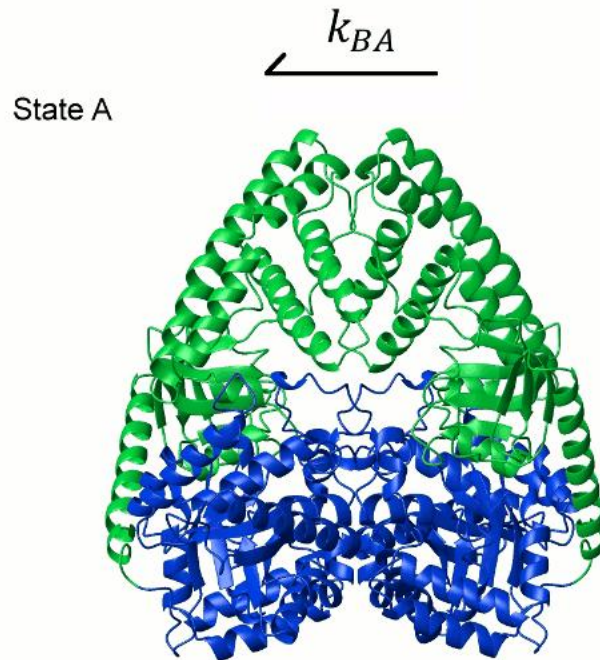
Spectral density

$$J_l(\omega) = \text{Re} \left\{ \tilde{C}_l(\sigma) \Big|_{\sigma \rightarrow i\omega} \right\}$$

Experimental observables: R1, R2 etc.

# Illustrative calculations

EI dimer



Estimations of XplorNIH @ 300 K

$$\mathfrak{D}_x^A = 29.16 \times 10^7 [s^{-1}] \quad \tau_{\mathfrak{D}}^A = 53.73 [ns]$$

$$\mathfrak{D}_y^A = 31.47 \times 10^7 [s^{-1}]$$

$$\mathfrak{D}_z^A = 32.43 \times 10^7 [s^{-1}]$$

$$\mathfrak{D}_x^B = 15.71 \times 10^7 [s^{-1}] \quad \tau_{\mathfrak{D}}^B = 79.99 [ns]$$

$$\mathfrak{D}_y^B = 15.82 \times 10^7 [s^{-1}]$$

$$\mathfrak{D}_z^B = 30.99 \times 10^7 [s^{-1}]$$

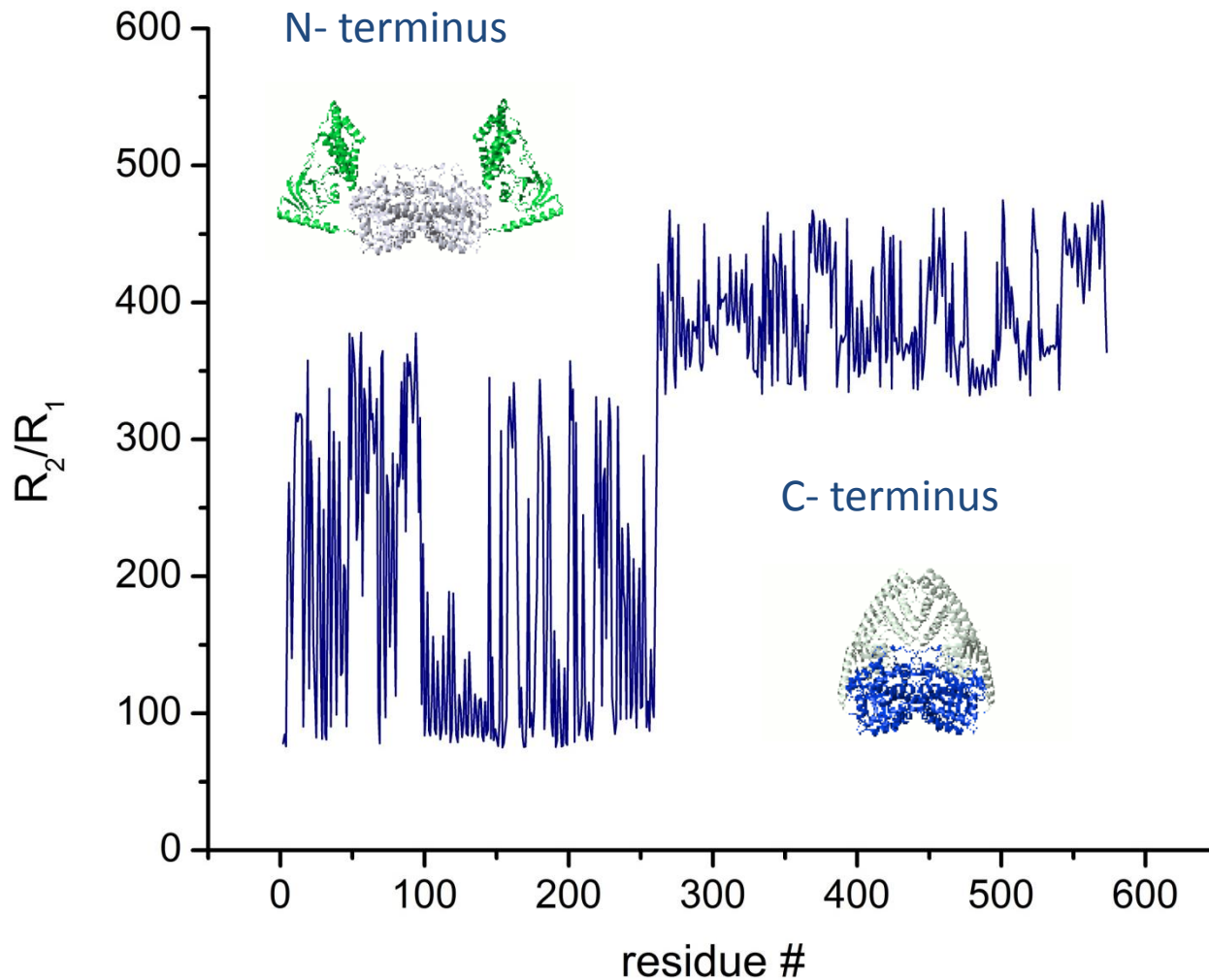
Assumptions

Symmetric motions  $\Omega_{AB}: \{\alpha_{AB} = 0, \beta_{AB} = 0, \gamma_{AB} = 0\}$

Equal occupation  $P_{eq}^A = P_{eq}^B = 1/2 \quad k_{AB} = k_{BA}$

## Illustrative calculations

NMR relaxation rates for 600 MHz @ 300 K

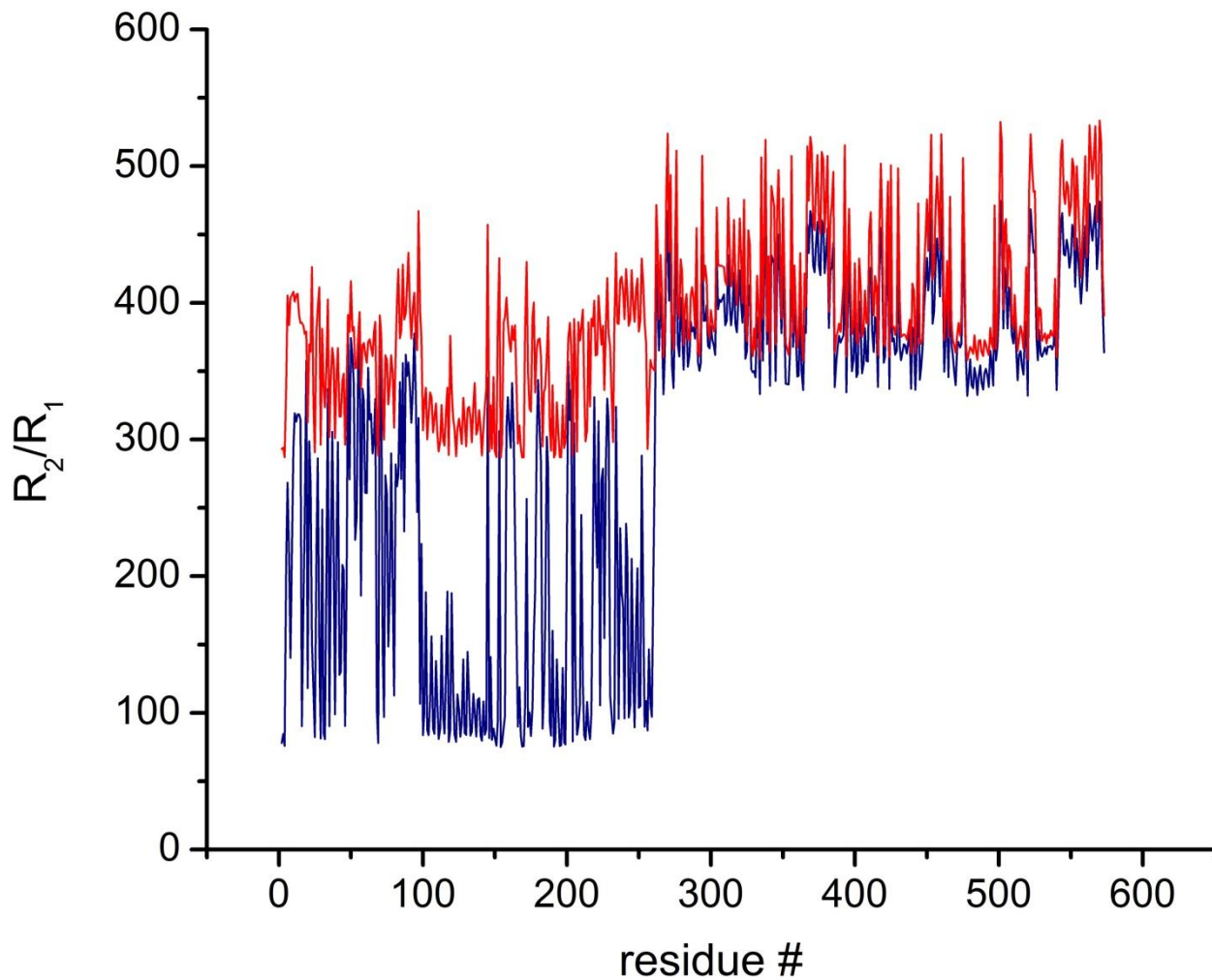


$$k = \frac{2}{\tau_D^A + \tau_D^B} = 155.57 \text{ [ns}^{-1}\text{]}$$



## Illustrative calculations

NMR relaxation rates for 600 MHz @ 300 K

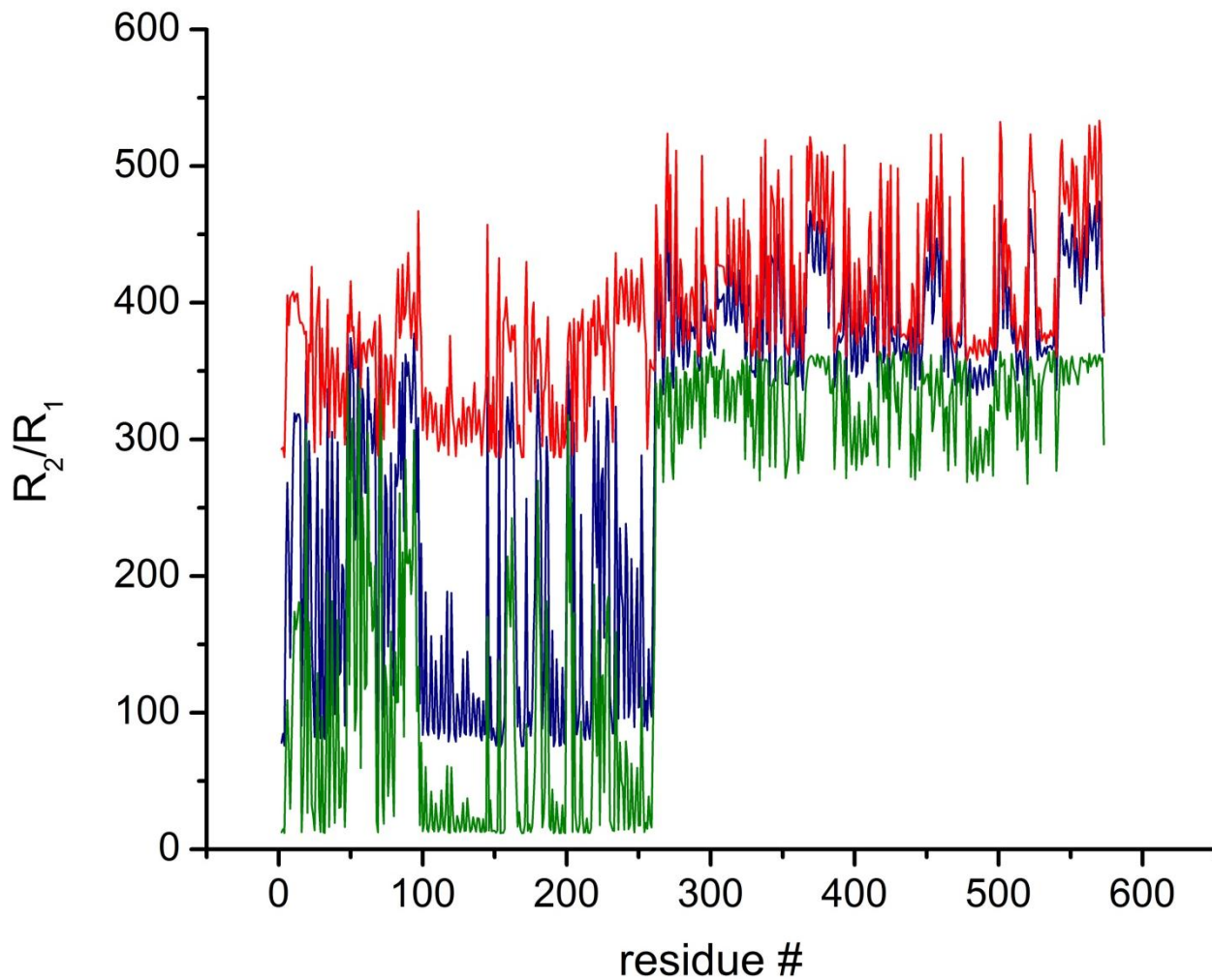


$$k_{slow} = 0.1 \times k$$

$$k = \frac{2}{\tau_D^A + \tau_D^B} = 155.57 \text{ [ns}^{-1}\text{]}$$

## Illustrative calculations

NMR relaxation rates for 600 MHz @ 300 K



$$k_{slow} = 0.1 \times k$$

$$k = \frac{2}{\tau_D^A + \tau_D^B} = 155.57 \text{ [ns}^{-1}\text{]}$$

$$k_{fast} = 10 \times k$$

## Concluding notes

### Our Model

- Provides closed form solutions in frequency domain ready for evaluation of spectral density etc.
- Provides known limiting cases and is reproduced by Monte Carlo simulations
- Not universal: discusses only the transitions between discrete states
- However, accounts for arbitrary symmetry of diffusion tensors, arbitrary reorientation of molecules upon conformation transition, and coupling between diffusion tumbling and conformation exchange

## Acknowledgments

### **Co-Authors**

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- G. Marius Clore

### **Extremely Useful Discussions**

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- Attila Szabo