Damped oscillations in view of the fractional oscillator equation

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This paper discusses the fractional oscillator equation involving fractional time derivatives of the Riemann-Liouville type. The exact solution of the fractional oscillator equation was obtained. On this basis the correspondence between the fractional time derivative and the dissipative properties was established. The relationship between the order of the fractional time derivative and the dissipative constant was derived. In addition to the exact solution, the perturbation approach was developed as well. It was shown that the perturbation method and exact solution lead to similar results. On the basis of the exact solution, the concept of fractional time evolution was illustrated.

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I. INTRODUCTION

These days the phenomena that obey the equation of motion with fractional derivatives have become a common theme. Introducing the fractional derivatives to describe physical phenomena seems to be beneficial for complex heterogeneous systems where conventional approaches have failed. As an example, one could mention the problem of anomalous diffusion.[1–4] Experimental examples of such transport are numerous, e.g., amorphous semiconductors,[5,6] polymers,[7–9] composite heterogeneous films,[10] porous media,[11,12] and many others (for references, see recent review Ref. 4). Today a phenomenon widely discussed in the framework of the so-called fractional diffusion approach implies that instead of using the first time derivative in the conventional transport equation, a time derivative of fractional order[4,9,13–19] may be used.

At the same time, similar features are inherent not only in the transport processes. Another generally acknowledged phenomenon described by the equation of motion with a fractional time derivative is the nonexponential relaxation of Cole-Cole type. Experimental evidence of this type of relaxation has preceded that of the anomalous transport,[20] although its relationships to the fractional derivatives were recognized only recently.[21–25] As well as the previous example, the nonexponential Cole-Cole relaxation is inherent in complex disordered materials such as associated liquids,[20] binary mixtures,[25,26] porous materials,[27] polymers,[28] polymer composites,[29] liquid crystals,[30] etc.

Despite the fact that introducing the fractional time derivatives in the cases mentioned above seems to be justified, there is no clear understanding of the basic reasons for fractional derivation in physics. From a mathematical point of view, the fractional calculus, which is a synonym of the term ''integration and differentiation of an arbitrary order,'' is a deeply elaborate and branched mathematical subject.[31–34] In general, introducing a fractional time derivation into differential equations that describe physical phenomena is justified in the framework of the fractional time evolution concept.[24,35,36] However, lack of physical applications leads to various problems. Many authors replace the integer time derivative by a fractional one on a purely mathematical or heuristic basis.[13–17,21–23] Moreover, the variety of mathematical definitions for the fractional derivatives[31–34] leads to the possibility to discuss different types of derivatives depending on the physical situation. Sometimes this raises conceptual difficulties in interpretation of the results even if one does not mention the issues related to the correct establishment of initial and border conditions.[19,24,33]

One possible way to clarify things is to discuss rather simple model problems involving fractional derivatives and to compare the results with classical analogs. In a sense, one may reiterate the way traversed by the conventional analysis to the physics. The mentioned examples of the fractional diffusion and nonexponential relaxation already clarify many of the aspects of the physical application of fractional calculus.[4,19,24,25] In this paper we will discuss the problem of the oscillator equation involving a fractional time derivative.

II. FRACTIONAL OSCILLATOR

In the present study we will be concerned with the equation

$$D_t^{2-\varepsilon}[x_\varepsilon(t)]+\nu_0^{2-\varepsilon}x_\varepsilon(t)=f_0\nu_0^{-\varepsilon}g_\varepsilon(t), \tag{1}$$

which we will call the ''fractional oscillator'' equation. In this equation $D_t^{2-\varepsilon}$ is the fractional Riemann-Liouville derivative operator of order $1\leqslant 2-\varepsilon<2$ and with a lower limit $t\to-\infty$ (see Appendix A).

Below we will compare Eq. (1) with the conventional equation of a damped oscillator:

$$\frac{d^2x_\tau(t)}{dt^2}+\frac{2}{\tau}\frac{dx_\tau(t)}{dt}+\omega_0^2x_\tau(t)=f_0g_\tau(t). \tag{2}$$

In Eqs. (1) and (2), the subscripts ε and τ are attributed to the fractional and conventional oscillator models, respectively, $x(t)$ is an unknown function that in the simplest case could be regarded as a displacement from the equilibrium position, t is a time variable, ω_0 and ν_0 are vibration eigenfrequencies, and the term proportional to $2/\tau$ describes the decay of oscillations in Eq. (2). The function $g(t)$ in the right-hand side of both equations is the so-called ''source function.'' Below, in order to provide energetically equivalent initial conditions for both oscillators, we will consider $g(t)$ as $g_\varepsilon(t)=\nu_0\delta(t)$ for Eq. (1) and $g_\tau(t)=\omega_0\delta(t)$ for Eq.

(2), where $\delta(t)$ is the Dirac δ function. This source function represents initial pulses of a force applied to the oscillators and f_0 is the strength of the pulses.

There are no doubts that Eq. (1) is not the only possible way to introduce the fractional derivatives into the oscillator equation. For instance, Ref. 37 discusses an approach based on the Caputo fractional derivative definition. Nevertheless, the choice of the Riemann-Liouville fractional derivative with lower limit $t\to-\infty$ is not arbitrary (see Appendix B). This choice provides us with the possibility to establish a physically reasonable initial condition and to discuss the Green's function of Eq. (1).

A. Naive approach

Using definition (B3), one can obtain a Fourier transform of the Green's functions for Eqs. (1) and (2) as

$$\tilde{x}_\varepsilon(\omega)=\frac{f_0\nu_0^{1-\varepsilon}}{(i\omega)^{2-\varepsilon}+\nu_0^{2-\varepsilon}} \tag{3}$$

and

$$\tilde{x}_\tau(\omega)=\frac{f_0\omega_0}{(i\omega)^2+2\,\tau^{-1}(i\omega)+\omega_0^2}. \tag{4}$$

Inverting the above equations, one can immediately get the solutions

$$x_\varepsilon(t)=f_0(\nu_0 t)^{1-\varepsilon}E_{2-\varepsilon,2-\varepsilon}[-(\nu_0 t)^{2-\varepsilon}] \tag{5}$$

for Eq. (1) and

$$x_\tau(t)=\frac{f_0\omega_0 e^{-t/\tau}}{\Omega}\sin(\Omega\, t) \tag{6}$$

for Eq. (2); where $\Omega=\sqrt{\omega_0^2-\tau^{-2}}$ and $E_{a,b}(y)$ is the Mittag-Leffler function (see Appendix A).

Note that both Eqs. (5) and (6) are physically valid only for $t\geqslant 0$, whereas for $-\infty<t<0$, $x(t)\equiv 0$. From a physical point of view, this fact is a consequence of the casuality principle. For Eq. (6), this directly follows from Eq. (4) that has poles only in the upper part of the ω complex plane. Thus, inverting Eq. (4) one obtains $x_\tau(t)\equiv 0$ for $t<0$. In contrast, for Eq. (3) in order to provide the casuality principle $x_\varepsilon(t)\equiv 0$ for $t<0$, one must establish the rule for bypassing of branching points in Eq. (3).

In Fig. 1, one can see an example of solutions (5) and (6). Qualitatively, the behavior of those solutions is similar. Both exhibit oscillations of $x(t)$ with decreased amplitude over time. Thus, let us compare those functions quantitatively. For this, let us calculate the spectral intensity $\tilde{x}^2(\omega)=\tilde{x}(\omega)\tilde{x}(\omega)^*$ for the Green's-function spectra (3) and (4),

$$\tilde{x}_\varepsilon^2(\omega)=\frac{f_0^2\nu_0^{2-2\varepsilon}}{[\omega^{2-\varepsilon}-\nu_0^{2-\varepsilon}\varphi_\varepsilon]^2+\nu_0^{4-2\varepsilon}[1-\varphi_\varepsilon^2]} \tag{7}$$

and

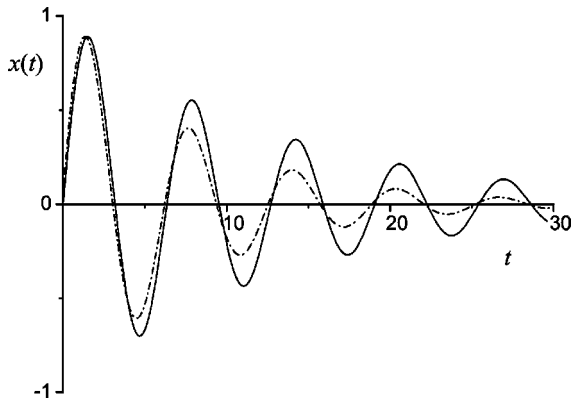

FIG. 1. Schematic picture presenting solutions of fractional [dash-dot line for $x_\varepsilon(t)$] and conventional [solid line for $x_\tau(t)$] oscillator equations. $\omega_0=\nu_0=1$, $f_0=1$, $\varepsilon=0.15$, and $\tau=6\frac{2}{3}$.

$$\tilde{x}_\tau^2(\omega)=\frac{f_0^2\omega_0^2}{[\omega^2-(\omega_0^2-2\,\tau^{-2})]^2+4\,\Omega^2\tau^{-2}}, \tag{8}$$

where $\varphi_\varepsilon=|\cos[\pi(2-\varepsilon)/2]|$, and the asterisk denotes complex conjugation.

In the above equations the terms $\nu_0^{2-\varepsilon}\varphi_\varepsilon$ and $\omega_0^2-2\tau^{-2}$ settle positions of the maxima of the spectral peaks while $\nu_0^{4-2\varepsilon}[1-\varphi_\varepsilon^2]$ and $4\,\Omega^2\tau^{-2}$ determine their height. Thus, comparing Eqs. (7) and (8), one could establish the following system of equations:

$$\nu_0\varphi_\varepsilon^{1/(2-\varepsilon)}=\sqrt{\omega_0^2-2\,\tau^{-2}},$$

$$\nu_0^2[1-\varphi_\varepsilon^2]^{1/(2-\varepsilon)}=\frac{2\,\Omega}{\tau}. \tag{9}$$

Resolving this system one finds the set of parameters ν_0, ω_0, ε, and τ for those spectral intensities (7) and (8) of the Green's functions (5) and (6) which have the same peak positions and height of their maxima.

Unfortunately, the exact solution of the transcendental system (9) is not obvious. Thus, let us discuss the situation where $\varepsilon\ll 1$ and $\tau^{-1}\ll\omega_0$; in other words, the situation when vibrations of the oscillators are weakly damped. After linearization, system (9) gives

$$\nu_0\cong\omega_0,$$

$$\varepsilon\cong\frac{4}{\pi\,\omega_0\tau}. \tag{10}$$

In Fig. 2, one can see the frequency dependence of the spectral intensity, real $\tilde{x}'(\omega)$ and imaginary $\tilde{x}''(\omega)$ parts of the Green's-function spectra (3) and (4), which obey relationship (10). The spectral intensity peak for the fractional oscillator is a little broader than the peak corresponding to the conventional oscillator. Thus, even for weak damping (by linear approximation), the total energy losses are larger in the case of the fractional oscillator. The difference between the behavior of fractional and conventional oscillators is even more clear for $\tilde{x}'(\omega)$ and $\tilde{x}''(\omega)$.

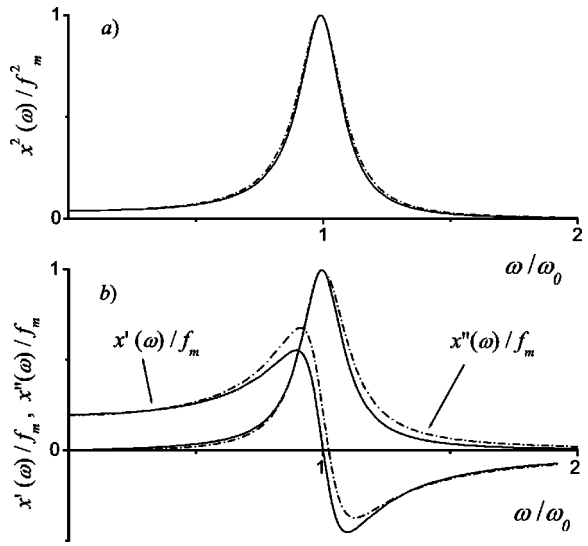

FIG. 2. (a) Spectral intensity normalized on its maximal value f_m^2. (b) Real and imaginary parts of the Green's functions normalized on f_m. On both figures, the dash-dot lines represent fractional oscillators and solid lines represent conventional oscillators. For all the curves, $\omega_0=\nu_0$, $\tau^{-1}=0.1\omega_0$, and $\varepsilon=4(\pi\,\omega_0\tau)^{-1}\approx 0.127$.

B. Perturbation theory

In general, the considerations presented above show that the presence of a fractional derivative in the oscillator equation is qualitatively equivalent to the presence of a dissipative term in the conventional oscillator equation. There are exact solutions for both models. However, the fractional oscillator equation as well as the equation of a damped oscillator are the idealized models of real systems. Other models involving a fractional time derivative could not imply exact solutions. Therefore, in other possible applications of the fractional calculus, one may be interested in an approximated solution obtained in terms of the perturbation theory. Thus, let us demonstrate the perturbation-theory technique in the case of a fractional oscillator.

The perturbation theory is a standard tool that is utilized in various fields of modern physics (see, for example, Ref. 38 on statistical mechanics). There are many ways to establish perturbation series. In Appendix C, we review basic definitions of the perturbation approach we used here.

Let us start with the equation of an oscillator without any energy losses, which we will regard as the unperturbed system,

$$\hat{F}_0[x(t)]=\frac{d^2x(t)}{d\,t^2}+\omega_0^2x(t)=f_0\delta(t), \tag{11}$$

where $\hat{F}_0$ is a differential operator corresponding to the unperturbed solution $x_0(t)=f_0\sin(\omega_0 t)$ for $t\geqslant 0$. For $-\infty<t<0$, $x_0(t)\equiv 0$ due to the casuality principle.

If there is a perturbation that can represented through the perturbation operator $\hat{F}_p$, then $x(t)$ obeys the following:

$$(\hat{F}_0+\hat{F}_p)[x(t)]=f_0\delta(t). \tag{12}$$

Suppose that the ratio of operator $\hat{F}_p$ and $\hat{F}_0$ norms is a small parameter $\|\hat{F}_0^{-1}\hat{F}_p\|\ll 1$. In this case, the solution $x(t)$ could be represented as a series of perturbation-theory corrections $x_k(t)$,

$$x(t)=\sum_{k=0}^{\infty}x_k(t). \tag{13}$$

The simplest way to calculate perturbation series in our case is to use the Fourier transform, which for the unperturbed operator gives

$$\tilde{F}_0=(i\omega)^2+\omega_0^2. \tag{14}$$

For a fractal oscillator equation, the perturbation operator may be represented as

$$\hat{F}_\varepsilon=\omega_0^\varepsilon D_t^{2-\varepsilon}-\frac{d^2}{d\,t^2}, \tag{15}$$

which after Fourier transform gives

$$\tilde{F}_\varepsilon=\omega_0^\varepsilon(i\omega)^{2-\varepsilon}-(i\omega)^2. \tag{16}$$

Then, in the case of a fractal oscillator following Eq. (C9), the Fourier transform of the first-order term is

$$\tilde{x}_{1,\varepsilon}(\omega)=-\frac{f_0\omega_0[\omega_0^\varepsilon(i\omega)^{2-\varepsilon}+\omega^2]}{(\omega_0^2-\omega^2)^2}. \tag{17}$$

The standard technique allows us to invert this expression and obtain for $\hat{F}_\varepsilon$ a first-order term in the perturbation series as

$$x_{1,\varepsilon}(t)=\frac{f_0}{2}\left[\omega_0 t\cos(\omega_0 t)+\sin(\omega_0 t)-\omega_0 t\cos\left(\omega_0 t-\frac{\pi\varepsilon}{2}\right)\right.$$
$$\left.-(1-\varepsilon)\sin\left(\omega_0 t-\frac{\pi\varepsilon}{2}\right)\right]. \tag{18}$$

For a conventional damped oscillator equation, the perturbation operator may be expressed as

$$\hat{F}_\tau=\frac{2}{\tau}\frac{d}{d\,t}, \tag{19}$$

which after Fourier transform gives

$$\tilde{F}_\tau=\frac{2\,i\,\omega}{\tau}. \tag{20}$$

Thus, following the procedure described above, one could obtain

$$\tilde{x}_{1,\tau}(\omega)=-\frac{2\,f_0\omega_0 i\omega}{\tau(\omega_0^2-\omega^2)^2}, \tag{21}$$

and after inverting,

$$x_{1,\tau}(t) = -\frac{f_0 t}{\tau}\sin(\omega_0 t). \tag{22}$$

It is interesting to note that the series expansion of Eq. (18) for $\varepsilon \ll 1$ gives $x_{1,\varepsilon}(t) \sim -(f_0 \pi\, \omega_0 t\, \varepsilon/4)\sin(\omega_0 t)$. Then, comparing this with Eq. (22), one could again obtain result (10). This proves that the developed perturbation approach is consistent with an exact solution (5).

Another observation is that the first-order correction of perturbation theory for a damped oscillator equation coincides with the first-order term of Taylor's expansion for exact solutions (6) with respect to the small parameter t/τ and the additional condition $\tau^{-1} \ll \omega_0$. However, in general, the terms of perturbation series do not necessarily coincide with Taylor's expansion terms because in the case of the perturbation approach described above, the role of small parameters belongs not to ε and $(\omega_0 \tau)^{-1}$, but to $\|\hat{F}_0^{-1}\hat{F}_\varepsilon\|$ and $\|\hat{F}_0^{-1}\hat{F}_\tau\|$.

C. Two bodies linked by a spring

The next model problem that could help us clarify physical features of a fractional oscillator is the problem of two bodies linked by a spring with dissipation (say two bodies situated in some viscose medium). Let us discuss two possibilities to introduce energy losses: through fractional time derivatives and by the conventional approach. We will only discuss a one dimensional problem and will assume that both bodies are equivalent. Thus, for these two models one could establish two systems of equations

$$D_t^{2-\varepsilon}[\,\eta_\varepsilon(t)] = \frac{f_0 \omega_0^{-\varepsilon}}{2} g_\varepsilon(t),$$

$$D_t^{2-\varepsilon}[\,\xi_\varepsilon(t)] + 2\,\omega_0^{2-\varepsilon}(\xi_\varepsilon(t) - \xi_0) = -f_0 \omega_0^{-\varepsilon} g_\varepsilon(t) \tag{23}$$

for the fractal oscillator and

$$\frac{d^2 \eta_\tau(t)}{d\,t^2} + \frac{2}{\tau}\frac{d\,\eta_\tau(t)}{d\,t} = \frac{f_0}{2} g_\tau(t),$$

$$\frac{d^2 \xi_\tau(t)}{d\,t^2} + \frac{2}{\tau}\frac{d\,\xi_\tau(t)}{d\,t} + 2\omega_0^2(\xi_\tau(t) - \xi_0) = -f_0 g_\tau(t) \tag{24}$$

for the conventional model.

In the above equations, $\eta = (y_2 + y_1)/2$ is the center of gravity position, $\xi = y_2 - y_1$ is the distance between the bodies' centers, ξ_0 is the distance between the bodies' centers at rest, and $f_0 g(t)$ once again represents the initial pulse of a force applied to the first body (see Fig. 3).

After Fourier transform for system (23), one gets

$$\tilde{\eta}_\varepsilon(\omega) = \frac{f_0}{2}\frac{\omega_0^{1-\varepsilon}}{(i\,\omega)^{2-\varepsilon}},$$

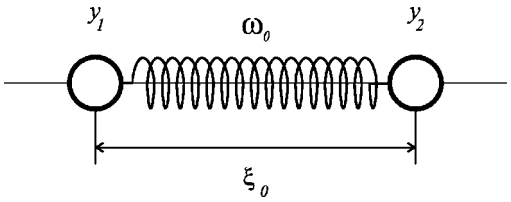

FIG. 3. Schematic picture explaining notations of Eqs. (23) and (24).

$$\tilde{\xi}_\varepsilon(\omega) = \frac{2\,\omega_0^{2-\varepsilon}\xi_0\delta_\omega(\omega) - f_0\omega_0^{1-\varepsilon}}{2\,\omega_0^{2-\varepsilon} + (i\omega)^{2-\varepsilon}}, \tag{25}$$

and

$$\tilde{\eta}_\tau(\omega) = \frac{f_0}{2}\frac{\omega_0}{[(i\,\omega)^2 + 2\,\tau^{-1}(i\,\omega)]},$$

$$\tilde{\xi}_\tau(\omega) = \frac{2\,\omega_0^2\xi_0\delta_\omega(\omega) - f_0\omega_0}{(i\omega)^2 + 2\,\tau^{-1}(i\,\omega) + 2\,\omega_0^2}, \tag{26}$$

for system (24). In those equations, $\delta_\omega(\omega)$ is once again the Dirac δ function, but in the frequency domain.

Using exact solutions (5) and (6) from Eqs. (25) and (26), one gets

$$\eta_\varepsilon(t) = \frac{f_0}{2\,\Gamma(2-\varepsilon)}(\omega_0 t)^{1-\varepsilon},$$

$$\xi_\varepsilon(t) = \xi_0 - f_0(\omega_0 t)^{1-\varepsilon}E_{2-\varepsilon,2-\varepsilon}[-2(\omega_0 t)^{2-\varepsilon}], \tag{27}$$

and

$$\eta_\tau(t) = \frac{f_0\omega_0\tau}{4}(1 - e^{-2\,t/\tau}),$$

$$\xi_\tau(t) = \xi_0 - \frac{f_0\omega_0 e^{-t/\tau}}{\sqrt{2\,\omega_0^2 - \tau^{-2}}}\sin(t\sqrt{2\,\omega_0^2 - \tau^{-2}}). \tag{28}$$

where $\eta(t) \equiv 0$ and $\xi(t) \equiv \xi_0$ for $-\infty < t < 0$ due to the casuality principle.

Figure 4 represents an example of solutions (27) and (28). From Fig. 4(b), one can see that the vibrations of fractional and conventional oscillators are qualitatively similar. For large energy losses ($\tau^{-1} = 0.3\,\omega_0$ and $\varepsilon \approx 0.4$) after the initial pulse, both oscillators perform a few vibrations and then return to equilibrium $\xi(t) \to \xi_0$. However, the behavior of the centers of gravity of those systems is qualitatively different [see Fig. 4(a)]. For the conventional oscillator, the evolution of the center gravity is finite, and at sufficiently long times reaches a stationary position $f_0\omega_0\tau/4$. The evolution of the center of gravity for the fractional oscillator is infinite and exhibits a power-law behavior.

III. CONCLUSIONS AND DISCUSSION

Let us summarize and discuss the results obtained. First, one can appreciate the exact solution (5) obtained for the

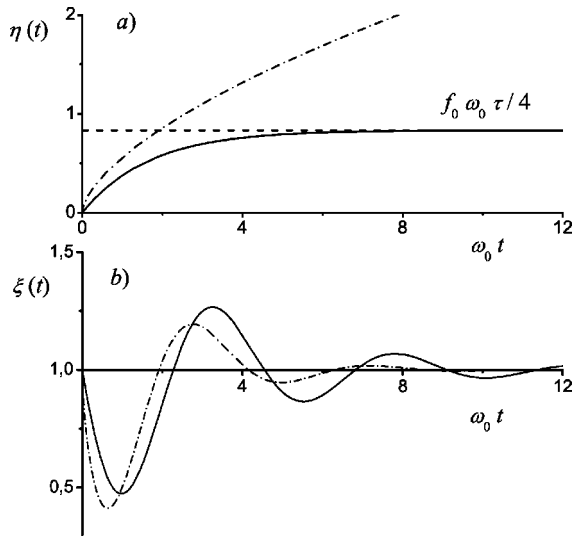

FIG. 4. Center of gravity position (a) and distance between the bodies' centers (b) as a function of time. On both figures, dash-dot lines represent the fractional oscillator and solid lines represent the conventional oscillator. For all the curves, $\tau^{-1}=0.3\,\omega_0$, $\varepsilon=4(\pi\,\omega_0\tau)^{-1}\approx 0.382$, and $f_0=\xi_0=1$.

fractional oscillator equation. In this regard, one must mention that the Mittag-Leffler function is commonly used in many physical problems to express a solutions of differential equations involving the fractional time derivatives.[22,24,37] For example, Ref. 22, on the basis of the fractal time random-walk relaxation model, discusses relaxation function proportional to the Mittag-Leffler function. In Ref. 24, the solution similar to Eq. (5) appears in relation to the fractional Cauchy problem and then later discussed as the probability density for a time interval between two consecutive jumps for the continuous time random walk scheme. The authors of Ref. 37 discuss a fractional oscillator model described by an integral equation involving a fractional integral. However, they assume the initial conditions in the same way as for the conventional damped oscillator. From our point of view, as we mentioned in this paper, this is not fully correct. Thus, they obtain a solution similar to Eq. (5) but without the power-law factor $(\omega_0 t)^{2-\varepsilon}$. Therefore, in the best of our knowledge, until now nobody discussed the solution in form (5) in relation to the fractional oscillator problem.

Solution (5) led us to the conclusion that introducing a fractional time derivative of the Riemann-Liouville type into the oscillator equation is qualitatively equivalent to introducing a dissipative term. This observation is not unexpected. The works[4,9,13–19,25] on anomalous diffusion and nonexponential relaxation also support this point.

The physical interpretation implies that the Green's function represents a response to δ-like impact. Therefore, the Green's function is related to the complex susceptibility of the systems discussed. The results represented in Fig. 2(b) show that the complex Fourier image of the Green's function for a fractional oscillator exhibits a slight excess of the high-frequency wing. Thus, at least hypothetically, the fractional oscillator model could be utilized to explain nonuniform broadening in various applications of spectroscopic methods.

Relationships (10) establish a connection between the order of the fractional time derivative and the parameters of the conventional oscillator equation. The general heuristic merit of this result is the fact that in the framework of the model discussed, the order of the fractional derivative was related to the vibrational frequency ω_0 as well as to the decay constant τ^{-1}. The vibrational frequency is dependent on the properties of the system itself. For example, in the case discussed ω_0 is dependent on the spring elastic modulus and the masses of the bodies. The decay constant is dependent on the property of the medium where the system is situated. For instance, in the case discussed, τ could be related to the viscosity of the medium. Thus, Eqs. (10) show that the fractional time derivative in the oscillator equation could be regarded as a result of the interplay between the system itself and the surrounding medium.

In Sec. II B of the presented paper, the application of the perturbation approach to the problem of fractional oscillator is discussed. Using this approach one could again obtain the result (10). This proves the self-consistency of the methods used. At first glance it seems that the existence of an exact solution (5) depreciates the significance of the perturbation method. However, the presented perturbation approach could be used not only for the problem of a fractional oscillator but also for other problems involving fractional time derivatives.

In Sec. II C, the problem of two bodies linked by a spring was discussed. This problem illustrates two significant points in the application of the fractional time derivatives to physical problems. First of all, it once again highlights the relationships between the fractional time derivative and the dissipative effects. Second, it shows that the fractional time derivatives lead to the possibility of a new class of stationarity. This fractional stationary was mentioned in the framework of the so-called fractional time evolution concept.[24,35,36] This concept establishes in addition to the conventional constants a second class of stationary states that obey the power-law time dependence. In Fig. 4(a), one can see an illustration of this fact. After the initial pulse of a force, the center of gravity position of the conventional oscillator goes to a constant stationary position while the center of gravity of the fractional oscillator goes to infinity as $f_0(\omega_0 t)^{1-\varepsilon}$. The behavior of the fractional oscillator in this case also illustrates the concept "coarse graining" for degrees of freedom. The variables y_1 and y_2, independent at short times, later become coupled and exhibit joint power-law behavior.

There are no doubts that the models presented are simplified examples rather than models of real physical situations. However, we hope that these examples help clarify general ideas that underlay possible physical application of fractional time derivatives. Moreover, the presented models could be improved. For example, one could discuss nonharmonic vibrations and the problem of several linked fractional oscillators. In this case, the mentioned perturbation approach could be helpful.

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APPENDIX A: DEFINITIONS

The Riemann-Liouville fractional derivative of order $n-1 \leqslant \gamma < n$ is defined[31–34] as

$$ {}_aD_t^{\gamma}[f(t)] = \frac{d^n}{dt^n} {}_aI_t^{n-\gamma}[f(t)], \tag{A1} $$

where

$$ {}_aI_t^{\vartheta}[f(t)] = \frac{1}{\Gamma(\vartheta)} \int_a^t (t-t')^{\vartheta-1} f(t')\, dt' \tag{A2} $$

is the Riemann-Liouville fractional integral of order $0<\vartheta$ with lower limit $t \to a$, $-\infty \leqslant a < t < +\infty$, $n \in \mathbb{N}$, and $\Gamma(\vartheta)$ is the Γ function.

An alternative way to define a fractional derivative of the Riemann-Liouville type is

$$ {}_a\overline{D}_t^{\gamma}[f(t)] = {}_aI_t^{n-\gamma}\left[\frac{d^n}{dt^n} f(t)\right]. \tag{A3} $$

Then

$$ {}_aD_t^{\gamma}[f(t)] = {}_a\overline{D}_t^{\gamma}[f(t)] + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\gamma}}{\Gamma(1+k-\gamma)}, \tag{A4} $$

where

$$ f^{(k)}(a) = \left.\frac{d^k f(t)}{dt^k}\right|_{t \to a}. $$

Note that in this paper, instead of the usual way of writing ${}_{-\infty}D_t^{\gamma}$ for a fractional derivative of order γ with a lower limit $t \to -\infty$, we will use notation D_t^{γ} for simplification.

Sometimes solution of differential equations involving the fractional time derivatives may be represented through the so-called Mittag-Leffler function[39]

$$ E_{a,b}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(a\,k+b)}. \tag{A5} $$

APPENDIX B: INTEGRAL TRANSFORMS AND INITIAL CONDITIONS

Let us now discuss Laplace and Fourier transforms of the Riemann-Liouville fractional derivatives. Let us define the direct Laplace transform as

$$ \mathcal{L}[f(t)] = \tilde{f}_{\mathcal{L}}(s) = \int_0^{\infty} e^{-s\,t} f(t)\, dt \tag{B1a} $$

and inverse Laplace transform as

$$ \mathcal{L}^{-1}[\tilde{f}_{\mathcal{L}}(s)] = f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\,t} \tilde{f}_{\mathcal{L}}(s)\, ds, \tag{B1b} $$

with Laplace variable s, fixed $\sigma > 0$, and $i = \sqrt{-1}$. Then, from Eqs. (A1) and (B1a), for $a = 0$, one realizes[31–34] that

$$ \mathcal{L}[{}_0D_t^{\gamma}[f(t)]] = s^{\gamma} \tilde{f}_{\mathcal{L}}(s) - \sum_{k=0}^{n-1} s^k\, {}_0D_t^{\gamma-1-k}[f(t)]|_{t \to 0+}. \tag{B2} $$

Let

$$ \mathcal{F}[f(t)] = \tilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\,\omega t} f(t)\, dt \tag{B3a} $$

be the direct Fourier transform, and

$$ \mathcal{F}^{-1}[\tilde{f}(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\,\omega t} \tilde{f}(\omega)\, d\omega \tag{B3b} $$

be the inverse Fourier transform with cyclic frequency ω. Then, from Eqs. (A1), (A3), and (B3a) for $a \to -\infty$, one could obtain[31–34]

$$ \mathcal{F}[D_t^{\gamma}[f(t)]] = \mathcal{F}[\overline{D}_t^{\gamma}[f(t)]] = (i\omega)^{\gamma} \tilde{f}(\omega). \tag{B4} $$

Next, let us discuss a differential equation

$$ \hat{F}[f(t)] = 0, \tag{B5} $$

where $\hat{F}$ is a differential operator involving the fractional time derivatives of the highest-order ${}_0D_t^{\gamma}$ with $n-1 \leqslant \gamma < n$ and solution $f(t)$ is defined for $t > 0$. In order to obtain this solution following Eq. (B2), one must establish n initial conditions in the form

$$ {}_0D_t^{\gamma-1-k}[f(t)]|_{t \to 0+} = C_k^{\gamma}, \quad k = 0,1,\ldots,n-1, \tag{B6} $$

where all C_k^{γ} are constants. For an integer γ, one easily recovers a conventional way to establish the initial conditions. For example, let us discuss $\hat{F}[f(t)] = \ddot{f}(t) + 2\,\tau^{-1}\dot{f}(t) + \omega_0^2 f(t)$, with $\gamma = 2$. Then, following Eq. (B6) in order to obtain $f(t)$, one needs to establish the initial position $f(0) = C_1^2$ and velocity $\dot{f}(0) = C_0^2$. However, for a fractional γ, the constants C_k^{γ} are hardly interpretable from the physical point of view. Thus, let us discuss an alternative approach.

It is known[40] that initial conditions for the homogeneous differential equation (B5) could be converted into generalized sources on the right-hand side of the nonhomogeneous equation

$$ \hat{F}[f(t)] = g(t), \tag{B7} $$

where the function $g(t)$ is dependent on the initial conditions themselves and the particular form of $\hat{F}$. In the general case, the function $g(t)$ could involve generalized functions such as the Dirac δ function $\delta(t)$ and the Heaviside step function $\theta(t)$. For example, if once again $\hat{F}[f(t)] = \ddot{f}(t) + 2\,\tau^{-1}\dot{f}(t)$

$+\omega_0^2 f(t)$, then the initial conditions $f(0)=C_1^2$ and $\dot{f}(0)=C_0^2$ could be converted into $g(t)=\theta(t)C_1^2+\delta(t)(C_0^2+2\,\tau^{-1}C_1^2)$.

However, for certain initial conditions (B6), in order to obtain the solution $f(t)$ of differential equation (B5) for $t>0$, one needs to establish $g(t)$ for $-\infty<t<+\infty$. More rigorously speaking, one needs to establish an analytic continuation of $\hat{F}$ and $f(t)$ for $-\infty<t\leqslant 0$. This can easily be done by replacing all fractional derivatives ${}_0D_t^\gamma$ with lower limit $t\to 0$ with fractional derivatives D_t^γ with lower limit $t\to-\infty$.

Then, the solution of the problem can be performed in the framework of the Green's-function approach. In this case, one must first resolve

$$\hat{F}[G(t)]=\delta(t), \tag{B8}$$

where $G(t)$ is the so-called Green's function of Eq. (B7). Then, using the convolution theorem, the solution $f(t)$ for any $g(t)$ could be obtained as

$$f(t)=\int_{-\infty}^{+\infty} G(t-t')g(t')\,dt'. \tag{B9}$$

In contrast to C_k^γ, a physical interpretation of $\delta(t)$ is more clear. For example, in Eqs. (1) and (2), it represents an initial pulse of a force and the Green's functions of those equations represent responses to this impact. Thus, due to Eq. (B9), any $g(t)$ could be regarded as the amplitude of the force that changes with time.

APPENDIX C: PERTURBATION SERIES

Let us discuss a physical system that obeys the differential equation

$$\hat{F}_0[x(t)]=g(t), \tag{C1}$$

where $\hat{F}_0$ is an unperturbed differential operator. Then, let us suppose that this physical system is exposed to the perturbation. Let $\hat{F}_p$ be the operator that describes this perturbation. Let us assume that both $\hat{F}_0$ and $\hat{F}_p$ are linear operators and $\|\hat{F}_0^{-1}\hat{F}_p\|\ll 1$, where $\|\cdots\|$ means the norm of the operator. In this case let us find the solution of the total equation

$$(\hat{F}_0+\hat{F}_p)[x(t)]=g(t) \tag{C2}$$

in the form

$$x(t)=\sum_{k=0}^{\infty} x_k(t). \tag{C3}$$

The first term of the perturbation series, which corresponds to the solution of an unperturbed equation (C1) is

$$x_0(t)=\hat{F}_0^{-1}[g(t)]. \tag{C4}$$

Then let us discuss the second term of the series (C3). In this case,

$$(\hat{F}_0+\hat{F}_p)[x_0(t)+x_1(t)]=\hat{F}_0[x_0(t)]+\hat{F}_0[x_1(t)]+\hat{F}_p[x_0(t)]+\hat{F}_p[x_1(t)]=g(t). \tag{C5}$$

Taking into account Eq. (C4) and neglecting the term $\hat{F}_p[x_1(t)]$ of the second order with respect to the perturbation, one could obtain

$$x_1(t)=-\hat{F}_0^{-1}\hat{F}_p\hat{F}_0^{-1}[g(t)]. \tag{C6}$$

The next term could be evaluated by a similar procedure

$$(\hat{F}_0+\hat{F}_p)[x_0(t)+x_1(t)+x_2(t)]=\hat{F}_0[x_0(t)]+\hat{F}_0[x_1(t)]+\hat{F}_0[x_2(t)]+\hat{F}_p[x_0(t)]+\hat{F}_p[x_1(t)]+\hat{F}_p[x_2(t)]=g(t). \tag{C7}$$

Taking into account Eqs. (C4) and (C6), and neglecting the term $\hat{F}_p[x_2(t)]$ of the third order with respect to the perturbation, one could realize

$$x_2(t)=\hat{F}_0^{-1}\hat{F}_p\hat{F}_0^{-1}\hat{F}_p\hat{F}_0^{-1}[g(t)]. \tag{C8}$$

Then, similar considerations allow us to write

$$x_n(t)=(-1)^n(\hat{F}_0^{-1}\hat{F}_p)^n\hat{F}_0^{-1}[g(t)]. \tag{C9}$$

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